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# The Motion of a Ship, as a Floating Rigid Body, in a Seaway

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# MOTION OF A SHIP...IN A SEAWAY

## ERRATA SHEET

| | | |
|---|---|---|
| p. 1 | l. 8 + 9 | Read "should be small oscillations, |
| p. 6 | l. 12 | " "Krylov." |
| p. 8 | l. 15 | "referred." |
| p. 13 | l. 1 | "pressure" |
| | l. 3 | $\theta_{21}$ and $\theta_{11}$ |
| | l. 6 | [illegible]$_1$ instead of $n_1$ |
| | l. 14 | $\theta_{11}\Psi_4(x,y,z)$ |
| p. 14 | l. 6 | "result" |
| p. 17 | l. 1 | read "$\phi'_1$ in (1.1[illegible])" |
| p. 18 | l. 21 | replace "where" by "[illegible]" |
| p. 19 | 2nd line of equation (1.23) replace | $2\phi g\theta_{31}$ by $2\rho g\theta_{31}$ |
| p. 22 | In equations (1.27) and (1.2[illegible]) [illegible] | |
| p. 23 | l. 2 | read "they" |
| | l. 21 | " "but" |
| p. 27 | l. 6 | insert the words "of the number" between "interactions" and "with" |
| p. 32 | l. 13 | read "that" |
| p. 43 | l. 6 | $V_j$ |
| p. 51 | l. 3 | $y = 0$ |
| p. [illegible] | l. [illegible] | $\chi_0(x,y,z)$ |
| p. 56 | l. 7 | read "satisfies" |
| p. 64 | l. 10 | insert "cos" between [illegible] and [illegible] |
| | l. 13 | read "then" instead of "where". |

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# THE MOTION OF A SHIP, AS A FLOATING RIGID BODY IN A SEAWAY

by

J. J. Stoker and A. S. Peters

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# THE MOTION OF A SHIP, AS A FLOATING RIGID BODY, IN A SEAWAY

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## 1. Introduction and summary.

The purpose of this report is to develop the mathematical theory for the motion of a ship, to be treated as a freely floating rigid body under the action of given external forces (a propeller thrust, for example), under the most general conditions compatible with a linear theory and the assumption of an infinite ocean. This of course requires the amplitude of the surface waves to be small and, in general, that the motion of the water should be small oscillations near its rest position of equilibrium; it also requires the ship to have the shape of a thin disk so that it may have a translatory motion with finite velocity and still create only small disturbances in the water. In addition, the motion of the ship itself must be assumed to consist of small oscillations relative to a uniform translation. Within these limitations, however, the theory to be presented is quite general in the sense that no arbitrary assumptions about the interaction of the ship with the water are made, nor about the character of the coupling between the different degrees of freedom of the ship, nor about the waves present on the surface of the sea: the combined system of ship and sea is treated by using the basic mathematical theory of the hydrodynamics of a non-turbulent perfect fluid.

For example, the theory presented here would make it possible to determine the motion of a ship under given forces which is started with arbitrary initial conditions in a sea subjected to given surface pressures and initial conditions, or in a sea covered with waves of prescribed character coming from infinity.

It is of course well known that such a linear theory for the non-turbulent motion of a perfect fluid, complicated though it is, still does not contain all of the important elements needed for a thoroughgoing discussion of the practical problems involved. For example, it ignores the boundary-layer effects, turbulent effects, the existence in general of a wake, and other important effects of a non linear character. Good discussions of these matters can be found in papers of Lunde and Wigley [6]* and Havlock [3]. Nevertheless, it seems clear that an approach to the problem of predicting mathematically the motion of ships in a seaway under quite general conditions is a worth-while enterprise, and that a start should be made with the problem even though it is recognized at the outset that all of the important physical factors can not be taken into account. In fact, the theory presented here leads at once to a number of important qualitative statements without the necessity of producing actual solutions - for example, we shall see that certain resonant frequencies appear quite naturally, and in addition that they can be calculated solely with reference to the mass distribution and the given shape of the hull of the ship. Interesting observations about the character of the coupling between the various degrees of freedom, and about

*Numbers in square brackets refer to the bibliography at the end of this report.

the nature of the interaction between the ship and the water, are also obtained simply by examining the equations which the theory yields.

In order to describe the theory and results to be worked out in later sections of this report, it is necessary to introduce our notation and to go somewhat into details. In Fig. 1.1 the disposition of two of the coordinate systems used is indicated. The

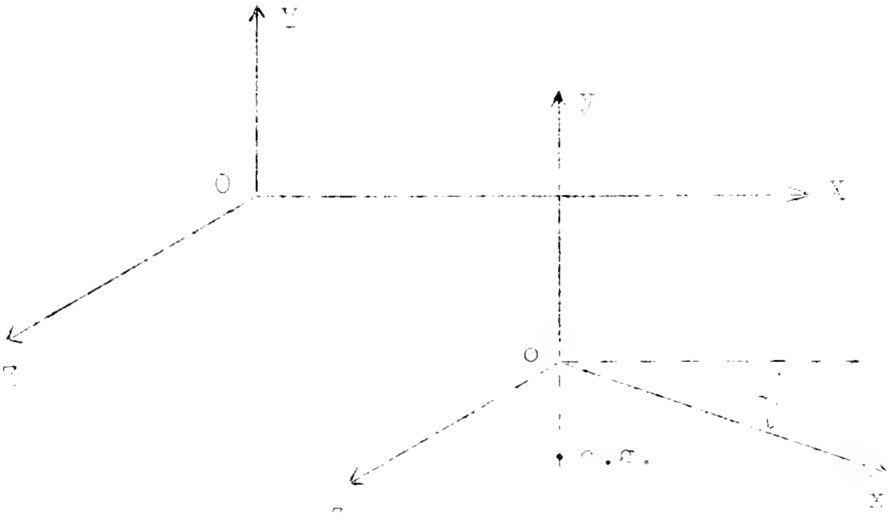



Fig. 1.1.

Fixed and Moving Coordinate Systems

system (X,Y,Z) is a system fixed in space with the X,Z-plane in the undisturbed free surface of the water and the Y-axis vertically upward.*

*This choice of axis is not the conventional one; the Z-axis is usually chosen as the vertical axis. It was made because the authors are accustomed to working with a variety of different water wave problems; and the choice made here seemed to them to be reasonable from a general point of view because of the large number of existing two-dimensional problems of interest in which one then naturally chooses the y-axis as vertical axis, coupled with the fact that the use of the symbol z for a complex variable is nearly universal.

A moving system of coordinates (x,y,z) is introduced; in this system the x,z-plane is assumed to coincide always with the X,Z-plane, and its y-axis is assumed to contain the center of gravity (abbreviated to c.g. in the following) of the ship. The course of the ship is fixed by the motion of the origin of the moving system; it is then convenient to introduce the speed s(t) of the ship in its course: the speed s(t) is simply the magnitude of the vector representing the instantaneous velocity of this point. At the same time we introduce the angular speed $\omega(t)$ of the moving system relative to the fixed system: the quantity fixes this rotation because the vertical axes remain always parallel. The angle $\alpha(t)$ indicated in Fig.1.1 is defined by

(1.1) $$\alpha(t) = \int_0^t \omega(t)dt .$$

In order to deal with the rigid body motion of the ship it is convenient, as always, to introduce a system of coordinates fixed in the body. Such a system (x',y',z') is indicated in Fig. 1.2

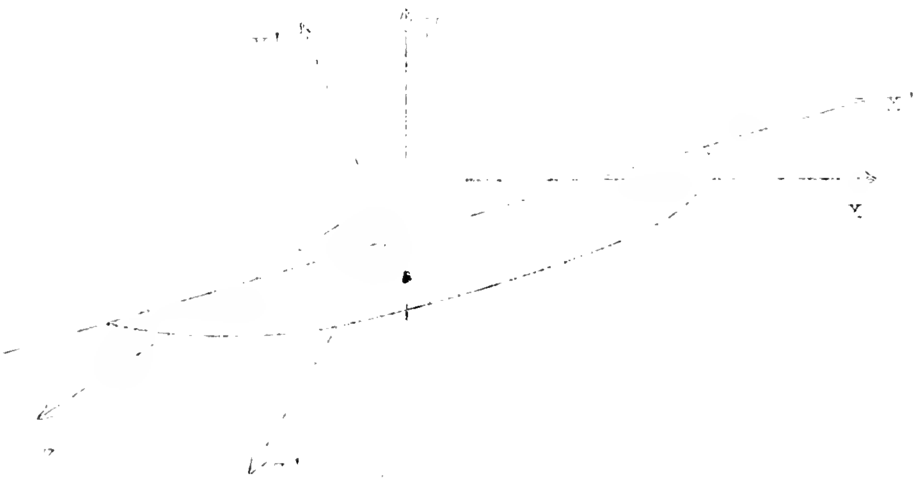


Fig. 1.2

The Moving Coordinate System

The $x',y'$-plane is assumed to be in the fore-and-aft plane of symmetry of the ship's hull, and the $y'$-axis is assumed to contain the c.g. of the ship. The moving system $(x',y',z')$ is assumed to coincide with the $(x,y,z)$ system when the ship and the water are at rest in their equilibrium positions. The c.g. of the ship will thus coincide with the origin of the $(x',y',z')$ system only in case it is at the level of the equilibrium water line on the ship, we therefore introduce the constant $y'_c$ as the coordinate of the c.g. in the primed coordinate system at such an instant.

The motion of the water is assumed to be given by a velocity potential* $\Phi(X,Y,Z;t)$; it in turn is therefore to be determined as a solution of Laplace's equation satisfying appropriate boundary conditions at the free surface of the water, on the hull of the ship, at infinity, and also initial conditions at the time $t = 0$. The boundary conditions on the hull of the ship clearly will depend on the motion of the ship, which in its turn is fixed, through the differential equations for the motion of a rigid body with six degrees of freedom, by the forces acting on it - including the pressure of the water - and its position and velocity at the time $t = 0$. As we have already stated, we make no further restrictive assumptions except those needed to linearize the problem.

Before discussing the linearization we interpolate a brief discussion of the relation of the present work to that of other writers who have discussed the problem of ship motions by means

*Thus it is implied that we deal with an irrotational motion of a non-viscous fluid.

of the linear theory of irrotational waves. The subject has a lengthy history, beginning with Michell [8] in 1898, and continuing over a long period of years in a sequence of notable papers by Havelock, beginning in 1909. This work is, of course, included as a special case in what is presented here. Extensive and up-to-date bibliographies can be found in the papers of Weinblum [20] and Lunde[7]. Most of this work considers the ship to be held fixed in space while the water streams past; the question of interest is then the calculation of the wave resistance in its dependence on the form of the ship. Of particular interest to us here are papers of Krylov [5], Weinblum and St. Denis [9], and Haskind [1], all of whom deal with less restricted types of motion. Krylov seeks the motion of the ship on the assumption that the pressure on its hull is fixed by the prescribed motion of the water without reference to the back effect on the motion of the water induced by the motion of the ship. Weinblum and St. Denis employ a combined theoretical and empirical approach to the problem which involves writing down equations of motion of the ship with coefficients which should be in part determined by model experiments; it is assumed in addition that there is no coupling between the different degrees of freedom involved in the general motion of the ship. Haskind attacks the problem in the same degree of generality, and under the same general assumptions, as the authors; in the end, however, Haskind derives his theory completely only in a certain special case. Haskind's theory is also not the same as

the theory presented here, and this is caused by a fundamental difference in the procedure used to derive the linear theory from the underlying, basically nonlinear, theory. Haskind develops his theory, in the time-honored way, by assuming that he knows a priori the relative orders of magnitude of the various quantities involved. Applied mathematicians are not often deceived in following such a procedure, but the present case is exceptional both because of its complexity and because of the fact that it is essential to consider terms which are not all of the same order. The authors also tried to attack the problem (without being aware at the time of the existence of Haskind's work) in this same way, but invariably arrived at formulations which seemed to be inconsistent. Consequently they felt it necessary to proceed by a formal development with respect to a small parameter (essentially the breadth-length ratio of the ship); in doing so every quantity was developed systematically in a formal series (for a similar type of discussion see F. John [4]). In this way a correct theory should be obtained, assuming the convergence of the series - and the authors see no reason to doubt that the series would converge for sufficiently small values of the parameter. Aside from the relative safety of such a method - purchased, it is true, at the price of making rather bulky calculations - it has an additional advantage, i.e., it makes possible a consistent procedure for determining any desired higher order corrections. It is not easy to compare Haskind's theory in detail with the theory presented here. However,

it can be stated that certain terms, called jumping terms by Haskind and considered to be of importance by him, are terms that would, in the theory presented here, be of higher order than any of those retained by the authors; consequently the authors feel that conclusions drawn from such terms may well be illusory unless some evidence is presented which shows these terms to be the most important among the very large number of different terms of that order which would occur in a formal development. A more precise statement on this point will be made later.

The procedure followed here begins by writing the equation of the ship's hull relative to the coordinate system fixed in the ship in the form

$$(1.2) \qquad z' = \pm \beta h(x',y') , \qquad z' \gtrless 0 \quad ,$$

with $\beta$ a small dimensionless parameter. This is the parameter referred to above with respect to which all quantities will be developed. In particular, the velocity potential $\Phi(X,Y,Z;t;\beta) \equiv \phi(x,y,z;t;\beta)$ is assumed to possess the development

$$(1.3) \quad \phi(x,y,z;t;\beta) = \beta\phi_1(x,y,z;t) + \beta^2\phi_2(x,y,z;t) + \cdots\cdots \quad .$$

The free surface elevation $\eta(x,z;t;\beta)$ and the speed $s(t;\beta)$ and angular velocity $\omega(t;\beta)$ (cf. 1.1) are assumed to have the developments

(1.4) $$\eta(x,z;t;\beta) = \beta\eta_1(x,z;t) + \beta^2\eta_2(x,z;t) + \cdots ,$$

(1.5) $$s(t;\beta) = s_o(t) + \beta s_1(t) + \cdots ,$$

(1.6) $$\omega(t;\beta) = \omega_o(t) + \beta\omega_1(t) + \cdots .$$

Finally, the vertical displacement $y_c(t)$ of the center of gravity and the angular displacements $\theta_1$, $\theta_2$, $\theta_3$ of the ship with respect to the x,y, and z axes respectively are assumed given by

(1.7) $$\theta_i(t;\beta) = \beta\,\theta_{i1}(t) + \beta^2\,\theta_{i2}(t) + \cdots,\quad i = 1,2,3,$$

(1.8) $$y_c(t;\beta) - y_c' = \beta y_1(t) + \beta^2 y_2(t) + \cdots .$$

These relations imply that the velocity of the water and the elevation of its free surface are small of the same order as the "slenderness parameter" $\beta$ of the ship. On the other hand, the speed s(t) of the ship is assumed to be of zero order. The other quantities fixing the motion of the ship are assumed to be of first order, except for $\omega(t)$, but it turns out in the end that $\omega_o(t)$ vanishes so that $\omega$ is also of first order. The quantity $y_c'$ in (1.8) was defined in connection with the description of Fig.1.2; it is to be noted that we have chosen to express all quantities with respect to the moving coordinate system (x,y,z) indicated in that figure. The formulas for changes of coordinates must be used, and they also are to be developed in powers of $\beta$; for example, the equation of the hull relative to the (x,y,z) coordinate system is found to be

$$z + \beta\theta_{21}x - \beta\theta_{11}(y-y_c') - \beta h(x,y) + \cdots = 0$$

after developing and rejecting second and higher order terms in $\beta$.

In marine engineering there is an accepted terminology for describing the motion of a ship; we wish to put it into relation with the notation just introduced. The angular displacements are named as follows: $\theta_1$ is the <u>rolling</u>, $\theta_2+\alpha$ is the <u>yawing</u>, and $\theta_3$ is the <u>pitching</u> oscillation. The quantity $s_1(t)$ in (1.5) is called the <u>surge</u> (i.e., it is the small fore-and-aft motion relative to the finite speed $s_0(t)$ of the ship), while $y_c$ fixes the <u>heave</u>. In addition, there is the side-wise displacement (in first order it might be denoted by $\beta z_1(t)$) referred to as the <u>sway</u>; this quantity, in lowest order, can be calculated in terms of $s_0(t)$ and the angle $\alpha$ defined by (1.1) in terms of $\omega(t)$ as follows:

$$\beta\dot{z}_1(t) = s_0\alpha = \beta s_0 \int_0^t \omega_1(t)dt, \tag{1.9}$$

since $\omega_c(t)$ turns out to vanish. In one of the problems of most practical interest, i.e. the problem of a ship that has been moving for a long time (so that all transients have disappeared) under a constant propeller thrust (considered to be simply a force of constant magnitude parallel to the keel of the ship) into a seaway consisting of a given system of simple harmonic progressing waves of given frequency, one expects that the displacement components would in general be the sum of two terms, one independent of the time and representing the displacements that would arise from

motion with uniform velocity through calm sea, the other a term simple harmonic in the time that has its origin in the forces arising from the waves coming from infinity. On account of the symmetry of the hull only two displacements of the first category would differ from zero: one in the vertical displacement, i.e. the heave, the other in the pitching angle, i.e. the angle $\theta_3$. The latter two displacements apparently are referred to as the trim of the ship. In all, then there would be in this case nine quantities to be fixed as far as the motion of the ship is concerned: the amplitudes of the oscillations in each of the six degrees of freedom, the speed $s_0$, and the two quantities determining the trim.

We proceed to give a summary of the theory obtained when the series (1.2) to (1.8) are inserted in all of the equations fixing the motion of the system, which includes both the differential equations and the boundary conditions, and any functions involving $\beta$ are in turn developed in powers of $\beta$. For example, one needs to evaluate $\phi_x$ on the free surface $y = \eta$ in order to express the boundary conditions there; one calculates it as follows (using (1.3) and (1.4)):

$$(1.10)\quad \phi_x(x,\eta,z;t;\beta) = \beta[\phi_{1x}(x,0,z;t) + \eta\phi_{1xy}(x,0,z;t) + \cdots]$$

$$+ \beta^2[\phi_{2x} + \eta\phi_{2xy} + \cdots] + \cdots$$

$$= \beta\phi_{1x}(x,0,z;t) + \beta^2[\eta_1\phi_{1xy}(x,0,z;t) + \phi_{2x}(x,0,z;t)] + \cdots .$$

We observe the important fact - to which reference will be made later - that the coefficients of the powers of $\beta$ are evaluated at $y = 0$, i.e. at the undisturbed equilibrium position of the free surface of the water. The end result of such calculations, carried out in such a way as to include all terms of first order in $\beta$ is as follows: The differential equation for $\phi_1$ is, of course, the Laplace equation:

$$\phi_{1xx} + \phi_{1yy} + \phi_{1zz} = 0 \tag{1.11}$$

in the domain $y < 0$, i.e. the lower half-space, excluding the plane area A of the x,y-plane which is the orthogonal projection of the hull, in its equilibrium position, on the x,y-plane. The boundary conditions on $\phi_1$ are

$$\begin{cases} \phi_{1z} = s_c(h_x - \theta_{21}) + (\omega_1 + \theta_{21})x - \dot{\theta}_{11}(y - y_c'), & \text{on } A_+ \\ \phi_{1z} = -s_c(h_x + \theta_{21}) + (\omega_1 + \theta_{21})x - \dot{\theta}_{11}(y - y_c'), & \text{on } A_- \end{cases} \tag{1.12}$$

in which $A_+$ and $A_-$ refer to the two sides $z = 0_+$ and $z = 0_-$ of the plane disk A. The boundary conditions on the free surface are

$$\begin{cases} g\eta_1 + s_c\phi_{1x} - \phi_{1t} = 0 \\ \phi_{1y} - s_c\eta_{1x} + \eta_{1t} = 0 \end{cases} \quad \text{at } y = 0. \tag{1.13}$$

The first of these results from the condition that the pressure vanishes on the free surface, the second arises from the kinematic free surface condition. If $s_0, \omega_1, \phi_{21}$, and $\phi_{11}$ were known functions of t, these boundary conditions in conjunction with (1.11) and appropriate initial conditions would serve to determine the functions $\phi_1$ and $\eta_1$ uniquely; i.e. the velocity potential and the free surface elevation would be known. In any case, the function $\phi_1$ - which we repeat, fixes the lowest order term in the development of the velocity potential $\phi$ - could be in principle determined, because of the linearity of the problem, as a linear combination of harmonic functions $\psi_i$ having $s_0, \omega_1 + \theta_{21}, \theta_{21}$ and $\theta_{11}$ as time-dependent coefficients:

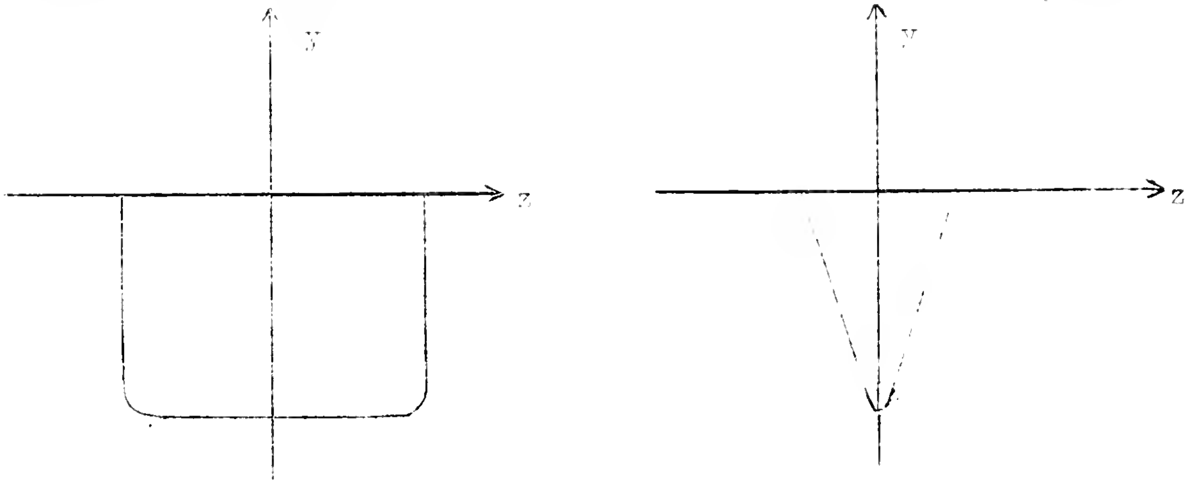
$$(1.14)\quad \phi_1(x,y,z;t) = s_0\psi_1(x,y,z) + (\omega_1 + \theta_{21})\psi_2(x,y,z)+\theta_{21}\psi_3(x,y,z)$$
$$+ \theta_{11}\psi_4(x,y,z) + \psi_5(x,y,z;t).$$

The harmonic function $\psi_5$ would be determined through initial conditions and the condition fixing the wave train coming from  -  that is, it contains the part of the solution arising from the non-homogeneous conditions in the problem.

Before continuing to describe the relations which determine the time-dependent coefficients in (1.14) as well as the other unknown functions of the time which fix the motion of the ship, we digress at this point in order to discuss some conclusions arising from our developments and concerning two questions which recur again and again in the literature. These issues center around the

question: what is the correct manner of satisfying the boundary conditions on the curved hull of the ship? Michell employed the condition (1.12), naturally with $\theta_{11} = \theta_{21} = \omega_1 \equiv 0$, on the basis of the physical argument that $s_0 h_x$ represents the component of the velocity of the water normal to the hull and since the hull is slender, a good approximation would result by using as boundary condition the jump condition furnished by (1.12). Havelock and others have usually followed the same practice. However, one finds constant criticism of the resulting theory in the literature (particularly the engineering literature) because of the fact that the boundary condition is not satisfied at the actual position of the ship's hull, and various proposals have been made to improve the approximation. The authors feel that this criticism is beside the point, since the condition (1.12) is simply the consequence of a reasonable linearization of the problem. To take account of the boundary condition at the actual position of the hull would, of course, be more accurate-- but then, it would be necessary to deal with the full nonlinear problem and make sure that all of the essential correction terms of a given order were obtained. In particular, it would be necessary to examine the higher order terms in the conditions at the free surface - after all, the conditions (1.13), which are also used by Michell and Havelock, are satisfied at $y = 0$ and not on the actual displaced position of the free surface. One way to obtain a more accurate theory would be, of course, to carry out the perturbation scheme outlined here to higher order terms.

Still another point has come up for frequent discussion (cf., for example, Lunde and Wigley [6]) with reference to the boundary condition on the hull. It is fairly common in the literature to refer to ships of Michell's type, by which is meant ships which are slender not only in the fore-and-aft direction, but which are also slender in the cross-sections at right angles to this direction (cf. Fig.1.3) so that $h_y$, in our notation, is small. Thus ships with a rather broad bottom (cf. Fig.1.3), or, as it is also put, with a full mid-section, are often considered as ships to which the present theory does not apply. It is true that $h_y$ may




(a) (b)

Fig. 1.3.

Ships with full and with narrow mid-sections

become rather large near the keel of the ship for certain types of cross-sections, but nevertheless the linearization carried out above should remain valid since all that is needed is that the ship should not create too large disturbances, and this condition is

guaranteed by taking a long, slender ship. (It might also be noted that $h_x$ occurs in our theory only under integral signs.) In fact, there are experimental results (cf. Havelock [3]) which indicate that the theory is just as accurate for ships with a full cross-section as it is for ships of Michell's type.

After this digression we return once more to the description of the equations which determine the motion of the ship, and which arise from developing the equations of motion with respect to $\beta$ and retaining only the terms of order $\beta$ and $\beta^2$. (We observe again that it is necessary to consider terms of both orders.) In doing so the mass $M$ of the ship is given by $M = M_1\beta$, with $M_1$ a constant, since we assume the average density of the ship to be finite and its volume is of course of order $\beta$. The moments of inertia are also of order $\beta$. The propeller thrust is assumed to be a force of magnitude $T$ acting in the negative $x'$-direction and in the $x', y'$-plane at a point whose $y'$-coordinate is $-\ell$; the thrust $T$ is of order $\beta^2$, since the mass is of order $\beta$ and accelerations are also of order $\beta$.

The terms of order $\beta$ yield the following conditions:

(1.15) $$[illegible] = 0 ,$$

(1.16) $$2\rho g \int_A \beta h \, dA = M_1 \beta g ,$$

(1.17) $$\int_A x\beta h \, dA = 0 ,$$

$$(1.18) \qquad \int_A [(\varphi_{1t} - s_0 \varphi_{1x})]^+_- \, dA = 0 \; ,$$

$$(1.19) \qquad \int_A [x(\varphi_{1t} - s_0 \varphi_{1x})]^+_- \, dA = 0 \; ,$$

$$(1.20) \qquad \int_A [y(\varphi_{1t} - s_0 \varphi_{1x}]^+_- \, dA = 0 \; .$$

The symbol $[\;]^+_-$ occurring here means that the jump in the quantity in brackets on going from the positive to the negative side of the projected area A of the ship's hull is to be taken. The variables of integration are x and y. The equation (1.15) states that the term of order zero in the speed is a constant, and hence the motion in the x-direction is a small oscillation relative to a motion with uniform velocity. Equation (1.16) is an expression of the law of Archimedes: the rest position of equilibrium must be such that the weight of the ship just equals the weight of the water it displaces. Equation (1.17) expresses another law of equilibrium of a floating body, i.e., that the center of buoyancy should be in the same vertical line as the center of gravity of the ship. The remaining three equations (1.18), (1.19), and (1.20) in the group serve to determine the displacements $\theta_{11}$, $\theta_{21}$, and $\omega_1$, which occur in the boundary condition (1.12) for the velocity potential $\varphi_1$. As we have already remarked, the velocity potential $\varphi_1$ can be written in the form (1.14) as a linear combination of harmonic functions with these unknown and time-dependent displacements as coefficients; insertion in equations

(1.18), (1.19), and (1.20) clearly lead to a coupled system of ordinary differential equations with constant coefficients for $\theta_{11}$, $\theta_{21}$, and $\omega_1$, which is of second order in $\theta_{11}, \theta_{21}$, and of first order in $\omega_1$ (though also of second order in the angular displacement $\alpha_1 = \int_0^t \omega_1(t)dt$). The coefficients of these differential equations are, of course, obtained in terms of integrals over A which involve the known functions $\psi_i(x,y,0_\pm,t)$. If the speed $s_c$ = const. (which occurs in (1.14)) is known, it follows that the system of differential equations for $\theta_{11}(t)$, $\theta_{21}(t)$, and $\omega_1(t)$ would yield these displacements uniquely once proper initial conditions are prescribed. We shall see in a moment that $s_c$ is fixed by a condition that is independent of all the unknown displacements - in fact, it depends only on the propeller thrust T and the shape of the hull - and consequently we have obtained a result that is at first sight rather surprising: the motion of the water, which is fixed solely by $\varphi_1$, is entirely independent of the pitching displacement $\theta_{31}(t)$, the heave $y_c(t)$, and the surge $s_1(t)$, i.e. of all displacements in the vertical plane except the constant forward speed $s_c$. A little reflection, however, makes this result quite plausible: Our theory is based on the assumption that the ship is a thin disk whose thickness is a quantity of first order disposed vertically in the water, hence only the finite displacements of the disk parallel to this vertical plane - create oscillations in the water that are of second order at least. On the other hand, displacements of first

order of the disk at right angles to itself will create motions in the water that are also of first order. One might seek to describe the situation crudely in the following fashion. Imagine a knife blade held vertically in the water. Up-and-down motions of the knife evidently produce motions of the water which are of a quite different order of magnitude from motions produced by displacements of the knife perpendicular to its blade. Stress is laid on this phenomenon here because it helps to promote understanding of other occurrences to be described later.

The terms of second order in $\beta$ yield, finally, the following conditions:

$$(1.21)\qquad M_1\dot{s}_1 = \rho\int_A [h_x(\varphi_{1t} - s_0\varphi_{1x})]^+_- \, dA + T,$$

$$(1.22)\qquad M_1\ddot{y}_1 = -2\rho g\int_L (y_1 + x\theta_{31})h\,dx$$

$$+\,\rho\int_A \left\{(h_y+\theta_{11})(\varphi_{1t}-s_0\varphi_{1x})\big|^+ + (h_y-\theta_{11})(\varphi_{1t}-s_0\varphi_{1x})\big|^-\right\} dA,$$

$$(1.23)\qquad I_{31}\ddot{\theta}_{31} = -2\rho g\theta_{31}\int_A (y-y_c')h\,dA - 2\rho g y_1\int_L xh\,dx$$

$$-\,2\rho g\dot{\theta}_{31}\int_L x^2h\,dx + lT + \rho\int_A [xh_y-(y-y_c')h_x][\varphi_{1t}-s_0\varphi_{1x}]^+_- \, dA.$$

We note that integrals over the projected water-line $L$ of the ship in its equilibrium position occur in addition to integrals over the

vertical projection A of the entire hull. The quantity $I_{31}$ arises from the relation $I = \beta I_{31}$ for the moment of inertia I of the ship with respect to an axis through its c.g. parallel to the z'-axis. The equation (1.21) determines the surge $s_1$, and also the speed $s_o$ (or, if one wishes, the thrust T is determined if $s_o$ is assumed to be given). Furthermore, the speed $s_o$ is fixed solely by T and the geometry of the ship's hull. This can be seen, with reference to (1.14) and the discussion that accompanies it, in the following way. If $\omega_1$, $\theta_1$, and $\theta_2$ are constants, then they must, as one could show, be identically zero; hence the term $s_o\psi_1$ in (1.14) is the only term in $\phi_1$ that is independent of t. It therefore determines T upon insertion of $\phi_1$ in (1.21). This term, however, is obtained by determining the harmonic function $\psi_1$ as a solution of the boundary problem for $\phi_1$ in the special case in which $\dot{\theta}_{11} = \dot{\theta}_{21} = \omega_1 = 0$; hence, as one sees from (1.12) and (1.13), $\psi_1$ is fixed by $s_o$ and $h_x$ alone. In fact, the relation between $s_1$ and T is exactly the same relation as was obtained by Michell. (It will be written down later.) In other words, the wave resistance is now seen to depend only on the basic motion with uniform speed of the ship, and not at all on its small oscillations relative to that motion. If, then, effects on the wave resistance due to the oscillation of the ship are to be obtained from the theory, it will be necessary to take account of higher order terms in order to calculate them. Once the thrust T has been determined the equations (1.22) and (1.23) form a coupled system for the determination of $y_1$ and $\theta_{31}$, since $\phi_1$ and $\theta_{11}$ have presumably been

determined previously. However, it is not quite correct to say that the surge $s_1$, the heave $y_1$, and the pitching oscillation $\theta_{31}$ are not coupled with the roll, yaw and sway since the latter quantities enter into the determination of $\phi_1$. Thus our system is one in which there is a great deal of cross-coupling. It might also be noted that the trim, i.e. the constant values of $y_1$ and $\theta_{31}$ about which the oscillations occur are determined from (1.22) and (1.23) by the time-independent terms in these equations -- including for example, the moment lT of the thrust about the c.g.

We have now given the complete formulation of our problem, except for initial conditions and conditions at $\infty$. Before saying anything about methods for finding concrete solutions in specific cases, it has some point to mention a number of conclusions, in addition to those already given, which can be inferred from our equations without solving them. Consider, for example, the equations (1.22) and (1.23) for the heave $y_1$ and the pitching oscillation $\theta_{31}$, and make the assumption that the integral $\int_L xhdx = 0$ (which means that the horizontal section of the ship at the water line has its c.g. on the same vertical as that of the whole ship). If this condition is satisfied it is immediately seen that the oscillations $\theta_{31}$ and $y_1$ are not coupled. Furthermore, these equations are seen to have the form

$$\ddot{y}_1 + \lambda_1^2 y_1 = p(t) \tag{1.25}$$

$$\ddot{\theta}_{31} + \lambda_2^2 \theta_{31} = q(t) \tag{1.26}$$

with

$$\lambda_1^2 = \frac{2\rho g \int_L h\,dx}{M_1} , \tag{1.27}$$

$$\lambda_2^2 = \frac{2\rho g\left[ \int_A (y - y_c')\,h\,dA + \int_L x^2 h\,dx \right]}{I_{31}} . \tag{1.28}$$

It follows immediately that resonance is possible if $p(t)$ has a harmonic component of the form $A\cos(\lambda_1 t + B)$ or $q(t)$ a component of the form $A\cos(\lambda_2 t + B)$: in other words, one could expect exceptionally heavy oscillations if the speed of the ship and the sea way were to be such as to lead to forced oscillations having frequencies close to these values. One observes also that these resonant frequencies can be computed without reference to the motion of the sea or the ship: the quantities $\lambda_1, \lambda_2$ depend only on the shape of the hull.*

In spite of the fact that the linear theory presented here must be used with caution in relation to the actual practical problems

*The equation (1.27) can be interpreted in the following way: it furnishes the frequency of free vibration of a system with one degree of freedom in which the restoring force is proportional to the weight of water displaced by a cylinder of cross-section area $2\int_L h\,dx$ when it is immersed vertically in water to a depth $y_1$.

concerning ships in motion, it still seems likely that such resonant frequencies would be significant if they happened to occur in the terms $p(t)$ or $q(t)$ with appreciable amplitudes. Suppose, for instance that the ship is moving in a sea-way that consists of a single train of simple harmonic progressing plane waves with circular frequency $\sigma$ which have their crests at right angles to the course of the ship. If the speed of the ship is s one finds that the circular excitation frequency of the disturbances caused by such waves, as viewed from the moving coordinate system $(x,y,z)$ that is used in the discussion here, is $\sigma + \frac{s\sigma^2}{g}$, since $\frac{\sigma^2}{g}$ is $2\pi$ times the reciprocal of the wave length of the wave train. Thus if $\lambda_1$ and $\lambda_2$ should happen to lie near this value, a heavy oscillation might be expected. One can also see that a change of course to one quartering the waves at angle $\gamma$ would lead to a circular excitation frequency $\sigma + s \cos\gamma \cdot \frac{\sigma^2}{g}$ and naturally this would have an effect on the amplitudes of the response.

It has already been stated that the work presented here is related to work published by Haskind [1] in 1946, and it was indicated that the two theories differ in some respects. We have not made a comparison of the two theories in the general case, which would not be easy to do, but it is possible to make a comparison rather easily in the special case treated by Haskind in detail. This is the special case treated in the second of his two papers in which the ship is assumed to oscillate only in the vertical

plane - as would be possible if the sea-way consisted in trains of plane waves all having their crests at right angles to the course of the ship. Thus only the quantities $s_1(t)$, $y_1(t)$, and $\theta_{31}(t)$, in our notation, would figure in fixing the motion of the ship. Haskind treats only the displacements $y_1(t)$ and $\theta_{31}(t)$ (which are denoted in his paper by $\zeta(t)$ and $\psi(t)$), for which he finds differential equations of second order; but these equations are not the same as the corresponding equations (1.22), (1.23) above. One observes that (1.22) contains as its only derivative the second derivative $\ddot{y}_1$ and (1.23) contains as its sole derivative a term with $\ddot{\theta}_{31}$; in other words there are no first derivative terms at all, and the coupling arises solely through the undifferentiated terms. Haskind's equations are quite different since first and second derivatives of both dependent functions occur in both of the two equations; thus Haskind, on the basis of his theory, can speak, for example, of damping terms, while the theory presented here does not yield any such terms. The authors feel that there should not be any damping terms of this order for the following reasons: In the absence of frictional resistances, the only way in which energy can be dissipated is through the transport of energy to infinity by means of out-going progressing waves. However, we have already given what seem to us to be valid reasons for believing that the oscillations that consist solely in displacements parallel to the vertical plane produce waves in the water with amplitudes that are of higher order than those considered

in the first approximation. Thus no such dissipation of energy would occur. In any case, our theory has this fact as one of its consequences. Of course, it does not matter too much if some terms of higher order are included in a perturbation theory, at least if all the terms of a certain given order are really present: at most, one might be deceived in giving too much significance to the higher order terms. Haskind also says, however, and we quote from the translation of his paper (see page 59): "Thus, for a ship symmetric with respect to its midship section ......., only in the absence of translatory motion, i.e. for $S_o = 0$, are the heaving and pitching oscillations independent." This statement does not hold in our version of the theory. As one sees from (1.22) and (1.23) coupling occurs if, and only if $\int_L xh\,dx \neq 0$, whether $S_o$ vanishes or not. In addition, Haskind obtains no resonant frequencies in these displacements because of the presence of first-derivative terms in his equations; the authors feel that such resonant frequencies may well be an important feature of the problem.

We turn next to a brief discussion of methods of solving the problems formulated above. The difficulties are for the most part concentrated in the problem of determining the first approximation $\phi_1(x,y,z,t)$ to the velocity potential. The discussion above assumed that $\phi_1$ had in some way been determined in the form (1.14) by solving the boundary value problem posed by (1.11), (1.12), (1.13), and appropriate conditions at the time t=0 and at $\infty$. In general, an explicit solution of the problem for $\phi_1$ - in terms of an integral

representation, say - seems out of the question. In fact, as soon as rolling or yawing motions occur, explicit solutions are unlikely to be found. The best that the authors have been able to do so far in such cases has been to formulate an integral equation for the values of $\phi_1$ over the vertical projection A of the ship's hull; this method of attack, which looks possible and somewhat hopeful for numerical purposes since the motion of the ship requires the knowledge of $\phi_1$ only over the area A, is under investigation. However, if the motion of the ship is confined to a vertical plane, so that $\omega_1 = \theta_{11} = \theta_{21} = 0$, it is possible to solve the problem explicitly. This can be seen with reference to the boundary conditions (1.12) and (1.13) which in this case are identical with those of the classical theory of Michell and Havelock, and hence permit an explicit solution for $\phi_1$ which is given later on in section 4. After $\phi_1$ is determined, it can be inserted in (1.21), (1.22), and (1.23) to find the forward speed $U_0$ corresponding to the thrust T, the two quantities fixing the trim, and the surging, pitching, and heaving oscillations. In all, six quantities fixing the motion of the ship are determined.

The theory developed in this report is very general, and it therefore could be applied to the study of a wide variety of different problems. For example, the stability of the oscillations of a ship could be investigated on a rational-dynamical basis, rather than at present by assuming the water to remain at rest when the ship oscillates. It would be possible in principle to investigate theoretically how a ship would move with a given

rudder setting, and find the turning radius, angle of heel, etc. The problem of stabilization of a ship by gyroscopes or other devices could be attacked in a very general way: the dynamical equations for the stabilizers would simply be included in the formulation of the problem together with the forces arising from the interactions with the hull of the ship.

In Sec. 2 the general formulation of the problem is given; in Sec. 3 the details of the linearization process are carried out; and in Sec. 4 a solution of the problem is given for the case of purely vertical motion, including a verification of the fact that the wave resistance is given by the same formula as was found by Michell.

## 2. General formulation of the problem.

We derive here the basic theory for the motion of a floating rigid body through water of infinite depth. The water is assumed to be in motion as the result of the motion of the rigid body, and also because of disturbances at sea; the combined system consisting of the rigid body and the water is to be treated as an interaction in which the motion of the rigid body, for example, is determined through the pressure forces exerted by the water on its surface. We assume that a velocity potential exists. Since we deal with a moving rigid body it is convenient to refer the motion to various types of moving coordinate systems as well as to a fixed coordinate system. The fixed coordinate system is denoted by 0-X,Y,Z. The X,Z-plane is in the equilibrium position of the free surface of the water, and the Y-axis is positive

upwards. The first of the two moving coordinate systems we use (the second will be introduced later) is denoted by $G$-$x,y,z$, and is specified by Fig. 1.1. The $x,z$-plane coincides with the $X,Z$-plane (i.e., it lies in the undisturbed free surface), the $y$-axis is vertically upward and contains the center of gravity of the ship. The $x$-axis has always the direction of the horizontal component of the velocity of the center of gravity of the ship. (If we define the course of the ship as the vertical projection of the path of its center of gravity on the $X,Z$-plane, then our convention about the $x$-axis means that this axis is taken tangent to the ship's course.) Thus if $R_c = (X_c, Y_c, Z_c)$ is the position vector of the center of gravity of the ship relative to the fixed coordinate system and hence $\dot{\vec{R}}_c = (\dot{X}_c, \dot{Y}_c, \dot{Z}_c)$ is the velocity of the c.g., it follows that the $x$-axis has the direction of the vector $\vec{u}$ given by

$$\vec{u} = \dot{X}_c \vec{I} + \dot{Z}_c \vec{J} \tag{2.1}$$

with $\vec{I}$ and $\vec{J}$ unit vectors along the $X$-axis and the $Z$-axis. If $\vec{i}$ is a unit vector along the $x$-axis we may write

$$s(t)\vec{i} = \vec{u} \quad , \tag{2.2}$$

thus introducing the speed $s(t)$ of the ship. For later purposes we also introduce the angular velocity vector $\vec{\omega}$ of the moving coordinate system:

$$(2.3) \qquad \vec{\omega} = \omega(t)\vec{J} \;,$$

and the angle $\alpha$ (cf. Fig. 1.1) by

$$(2.4) \qquad \alpha(t) = \int_0^t \omega(\tau)d\tau \;.$$

The equations of transformation from one coordinate system to the other are

$$(2.5) \quad \begin{cases} X = x \cos\alpha + z \sin\alpha + X_c \;; & x = (X-X_c)\cos\alpha - (Z-Z_c)\sin\alpha \\ Y = y \;; & y = Y \\ Z = -x \sin\alpha + z \cos\alpha + Z_c \;; & z = (X-X_c)\sin\alpha + (Z-Z_c)\cos\alpha \end{cases}$$

By $\Phi(X,Y,Z;t)$ we denote the velocity potential and write

$$(2.6) \quad \Phi(X,Y,Z,t) = \Phi(x\cos\alpha + z\sin\alpha + X_c, y, -x\sin\alpha + z\cos\alpha + Z_c; t) \equiv \phi(x,y,z;t) \;.$$

In addition to the transformation formulas for the coordinates, we also need the formulas for the transformation of various derivatives. One finds without difficulty the following formulas:

$$(2.7) \quad \begin{cases} \Phi_X = \phi_x \cos\alpha + \phi_z \cos\alpha \\ \Phi_Y = \phi_y \\ \Phi_Z = -\phi_x \sin\alpha + \phi_z \cos\alpha \;. \end{cases}$$

It is clear that $\text{grad}^2\, \Phi(X,Y,Z,t) = \text{grad}^2\, \phi(x,y,z;t)$ and that $\phi$ is a harmonic function in $x,y,z$ since $\Phi$ is harmonic in $X,Y,Z$. To calculate $\Phi_t$ is a little more difficult; the result is

$$(2.8) \qquad \Phi_t = -(s+\omega z)\phi_x + \omega x\phi_z + \phi_t \quad .$$

(To verify this formula, one uses $\Phi_t = \phi_x x_t + \phi_y y_t + \phi_z z_t + \phi_t$ and the relations (2.5) together with $s \cos \alpha = \dot{X}_c$, $s \sin \alpha = -\dot{Z}_c$.) The last two sets of equations make it possible to express Bernoulli's law in terms of $\phi(x,y,zt)$;

$$(2.9) \qquad \frac{p}{\rho} + gy + \frac{1}{2}(\text{grad}\, \phi)^2 + (s+\omega z)\phi_x - \omega x\phi_z - \phi_t = 0 \ .$$

Suppose now that $F(X,Y,Z;t) = 0$, is a boundary surface (fixed or moving) and set

$$(2.10) \qquad F(x \cos \alpha+\ldots,y,-x \sin \alpha+\ldots;t) \equiv f(x,y,z;t) \quad ,$$

so that $f(x,y,z;t) = o$ is the equation of the boundary surface relative to the moving coordinate system. The kinematic condition to be satisfied on such a boundary surface is that the particle derivative $\frac{dF}{dt}$ vanishes, and this leads to the boundary condition

$$(2.11) \qquad \phi_x f_x + \phi_y f_y + \phi_z f_z = -(s+\omega z)f_x + \omega x f_z + f_t$$

relative to the moving coordinate system upon using the appropriate transformation formulas. In particular, if $y - \eta(x,z;t) = 0$ is the equation of the free surface of the water, the appropriate kinematic condition is

$$(2.12) \qquad -\phi_x\eta_x + \phi_y - \phi_z\eta_z = (s+\omega z)\eta_x - \omega x\eta_z - \eta_t$$

to be satisfied for $y = \eta$. (The dynamic free surface condition is of course obtained for $y = \eta$ from (2.9) by setting $p = 0$.)

We turn next to the derivation of the appropriate conditions, both kinematic and dynamic, on the ship's hull. To this end it is convenient to introduce another moving coordinate system $o' - x', y', z'$ which is rigidly attached to the ship. It is assumed that the hull of the ship has a vertical plane of symmetry (which also contains the center of gravity of the ship); we locate the $x', y'$-plane in it (cf. Fig 1.2) and suppose that the $y'$-axis contains the center of gravity. The $o'-x', y', z'$ system, like the other moving system, is supposed to coincide with the fixed system in the rest position of equilibrium at $t=0$. The center of gravity of the ship will thus be located at a definite point on the $y'$-axis, say at distance $y'_c$ from the origin $O'$; in other words, the system of coordinates attached rigidly to the ship is such that the center of gravity has the coordinates $(0, y'_c, 0)$.

In the present section we do not wish in general to carry out linearizations. However, since we shall in the end deal only with motions which involve small oscillations of the ship relative

to the first moving coordinate system o-x,y,z, it is convenient and saves time and space to suppose even at this point that the angular displacement of the ship relative to the o-x,y,z system is so small that it can be treated as a vector $\vec{\theta}$:

$$\vec{\theta} = \theta_1\vec{i} + \theta_2\vec{j} + \theta_3\vec{k} \quad . \tag{2.13}$$

The transformation formulas, correct up to the first order terms in the components $\theta_i$ of $\vec{\theta}$, are then given by:

$$\begin{aligned} x' &= x + \theta_3(y-y_c) - \theta_2 z \\ y' &= y - (y_c-y_c') + \theta_1 z - \theta_3 x \\ z' &= z + \theta_2 x - \theta_1(y-y_c) \end{aligned} \tag{2.14}$$

with $y_c$ of course representing the y-coordinate of the center of gravity in the unprimed system. It is assumed that $y_c - y_c'$ is a small quantity of the same order as the quantities $\theta_i$ and only linear terms in this quantity have been retained. (The verification of (2.14) is readily carried out by making use of the vector-product formula $\vec{\delta} = \vec{\theta} \times \vec{r}$, for the small displacement $\vec{\delta}$ of a rigid body under a small rotation $\vec{\theta}$.)

The equation of the hull of the ship (assumed to be symmetrical with respect to the x',y'-plane) is now supposed given relative to the primed system of coordinates in the form:

$$(2.15)\qquad z' = \pm\, \zeta(x',y') \ , \qquad z' \gtrless 0 \ .$$

The equation of the hull relative to the o-x,y,z-system can be written in the form

$$(2.16)\quad z+\theta_2 x-\theta_1(y-y_c') - \zeta(x,y) + [\theta_2 z-\theta_3(y-y_c')]\zeta_x(x,y)$$

$$+[(y_c-y_c')-\theta_1 z+\theta_3 x]\zeta_y(x,y) = 0 \ , \ z' > 0 \ ,$$

when higher order terms in $(y_c-y_c')$ and $\theta_i$ are neglected. The left hand side of this equation could now be inserted for f in (2.11) to yield the kinematic boundary condition on the hull of the ship, but we postpone this step until the next section.

The dynamical conditions on the ship's hull are obtained from the assumption that the ship is a rigid body in motion under the action of the propeller thrust $\vec{T}$, its weight $-Mg\vec{j}$, and the pressure p of the water on its hull. The principle of the motion of the center of gravity yields the equation

$$(2.17)\qquad M\frac{d}{dt}(s\vec{i} + \dot{y}_c\vec{j}) = \int_S p\,\vec{n}\,dS + \vec{T} - Mg\vec{j} \ .$$

By $\vec{n}$ we mean the inward unit normal to the hull. Our moving coordinate system o-x,y,z is such that $\frac{d\vec{i}}{dt} = -\omega\vec{k}$ and $\frac{d\vec{j}}{dt} = 0$, so that (2.17) can be written in the form

$$(2.18) \qquad M\dot{s}\vec{i} - Ms\omega\vec{k} + M\ddot{y}_c\vec{j} = \int_S p\,\vec{n}\,dS + \vec{T} - Mg\vec{j} \quad ,$$

with p defined by (2.9). The law of conservation of angular momentum is taken in the form

$$(2.19) \qquad \frac{d}{dt}\int_M (\vec{R}-\vec{R}_c) \times (\dot{\vec{R}}-\dot{\vec{R}}_c)\,dm = \int_S p(\vec{R}-\vec{R}_c) \times \vec{n}\,dS + (\vec{R}_T-\vec{R}_c) \times \vec{T} \ .$$

The crosses all indicate vector products. By $\vec{R}$ is meant the position vector of the element of mass dm relative to the fixed coordinate system. $\vec{R}_c$ fixes the position of the c.g. and $\vec{R}_T$ locates the point of application of the propeller thrust T, also relative to the fixed coordinate system. We introduce $\vec{r} = (x, y, z)$ as the position vector of any point in the ship relative to the moving coordinate system and set

$$(2.20) \qquad \vec{q} = \vec{r} - y_c\vec{j} \quad ,$$

so that $\vec{q}$ is a vector from the c.g. to any point in the ship. The relation

$$(2.21) \qquad \dot{\vec{R}} = \dot{\vec{R}}_c + (\vec{\omega} + \dot{\bar{\omega}}) \times \vec{q}$$

holds, since $\vec{\omega} + \dot{\bar{\omega}}$ is the angular velocity of the ship, thus (2.21) is simply a statement of a basic kinematic property of rigid bodies. By using the last two relations the dimensional

condition (2.19) can be expressed in terms of quantities measured with respect to the moving coordinate system o-x,y,z, as follows:

$$(2.22)\qquad \frac{d}{dt}\int_M (\vec{r}-y_c\vec{j}) \times [(\vec{\omega}+\dot{\vec{\theta}}) \times (\vec{r}-y_c\vec{j})]\,dm = \int_S p(\vec{r}-y_c\vec{j}) \times \vec{n}\,dS + (\vec{R}_T - \vec{R}_c) \times \vec{T} .$$

We have now derived the basic equations for the motion of the ship. What would be wanted in general would be a velocity potential $\phi(x,y,z;t)$ satisfying (2.11) on the hull of the ship, conditions (2.9) (with $p = 0$) and (2.12) on the free surface of the water; and conditions (2.17) and (2.22), which involve $\phi$ under integral signs through the pressure $p$ as given by (2.9). In addition, there would be initial conditions and conditions at $\infty$ to be satisfied. Detailed consideration of these conditions, and the complete formulation of the problem of determining $\phi(x,y,z;t)$ under various conditions will be postponed, however until later on since we wish to carry out a linearization of all of the conditions formulated here.

3. Linearization by a formal perturbation procedure.

Because of the complicated nature of our conditions, it seems wise to carry out the linearization by a formal development in order to make sure that all terms of a given order are retained; this is all the more necessary since terms of different orders must be considered. The linearization carried out here is based

on the assumption that the motion of the water relative to the fixed coordinate system is a small oscillation about the rest position of equilibrium. It follows, in particular, that the elevation of the free surface of the water should be assumed to be small. We do not, however, wish to consider the speed of the ship with respect to the fixed coordinate system to be a small quantity: it should rather be considered a finite quantity. This brings with it the necessity to restrict the form of the ship so that its motion through the water does not cause disturbances so large as to violate our basic assumption; in other words, we must assume the ship to have the form of a thin disk. In addition, it is clear that the velocity of such a disk-like ship must of necessity maintain a direction that does not depart too much from the plane of the thin disk if only small disturbances in the water are to be created as a result of its motion with finite speed. Thus we assume that the equation of the ship's hull is given by

$$z' = \beta h(x',y') \quad , \quad z' > 0 \, , \tag{3.1}$$

with $\beta$ a small dimensionless parameter, so that the ship is a thin disk symmetrical with respect to the $x',y'$-plane, and $\beta h$ takes the place of $\zeta$ in (2.15). (It might be noted in passing that this is not the most general way to describe the shape of a disk that would be suitable for a linearization of the type carried out here.) We have already assumed in the preceding section that the motion of the ship is a small oscillation relative to the moving

coordinate system o-x,y,z - an assumption which, in fact, is made necessary by our basic assumptions concerning the linearization. It seems reasonable, therefore, to develop all of our basic quantities (taken as functions of x,y,z,t) in powers of $\beta$, as follows:

$$\phi(x,y,z;t;\beta) = \beta\phi_1 + \beta^2\phi_2 + \dots , \tag{3.2}$$

$$\eta(x,z;t;\beta) = \beta\eta_1 + \beta^2\eta_2 + \dots , \tag{3.3}$$

$$s(t;\beta) = s_0 + \beta s_1 + \beta^2 s_2 + \dots , \tag{3.4}$$

$$\omega(t;\beta) = \omega_0 + \beta\omega_1 + \beta^2\omega_2 + \dots , \tag{3.5}$$

$$\theta_i(t;\beta) = \beta\theta_{i1} + \beta^2\theta_{i2} + \dots , \tag{3.6}$$

$$y_c - y_c' = \beta y_1 + \beta^2 y_2 + \dots . \tag{3.7}$$

The first and second conditions state that the velocity potential and the surface wave amplitudes, as seen from the moving system, are small of order $\beta$. The speed of the ship, on the other hand, and the angular velocity of its c.g. about the vertical axis of the fixed coordinate system, are assumed to be of order zero. (It will turn out, however, that $\omega_0 = 0$ - a not unexpected result.) The relations (3.6) and (3.7) serve to make precise our previous assumption that the motion of the ship is a small oscillation relative to the system o-x,y,z.

We must now insert these developments in the conditions derived in the previous section. The free surface conditions are

treated first. As a preliminary step we observe that

$$(3.8)\quad \phi_x(x,\eta,z;t;\beta) = \beta[\phi_{1x}(x,0,z;t) + \eta\phi_{1xy}(x,0,z;t) + \dots]$$
$$+ \beta^2[\phi_{2x}+\eta\phi_{2xy}+ \dots]$$
$$+ \dots\dots\dots\dots\dots\dots$$
$$= \beta\phi_{1x}(x,0,z;t)+\beta^2[\eta_1\phi_{1xy}(x,0,z;t)+\phi_{2x}(x,0,z;t)]$$
$$+ \dots\dots\dots\dots\dots\dots \quad ,$$

with similar formulas for other quantities when they are evaluated on the free surface $y = \eta$ . Here we have used the fact that $\eta$ is small of order $\beta$ and have developed each term in a Taylor series. Consequently, the dynamic free surface condition for $y = \eta$ arising from (2.9) with $p = 0$ can be expressed in the form

$$(3.9)\quad \begin{aligned} &g[\beta\eta_1+\beta^2\eta_2+ \dots] + \tfrac{1}{2}\beta^2[(\operatorname{grad}\phi_1)^2 + \dots] \\ &+ [s_o+\beta s_1+\dots+z(\omega_o+\beta\omega_1+\dots)][\beta\phi_{1x}+\beta^2(\eta_1\phi_{1xy}+\phi_{2x})+\dots] \\ &- x(\omega_o+\beta\omega_1+\dots)[\beta\phi_{1z}+\beta^2(\eta_1\phi_{1zy}+\phi_{2z})+\dots] \\ &-[\beta\phi_{1t}+\beta^2(\eta_1\phi_{1ty}+\phi_{2t})+\dots] = 0 \end{aligned}$$

and this condition is to be satisfied for $y = 0$. In fact, as always in problems of small oscillations of continuous media, the boundary conditions are satisfied in general at the equilibrium

position of the boundaries. Upon equating the coefficient of the lowest order term to zero we obtain the dynamical free surface condition

$$(3.10)\qquad g\eta_1+(s_c+\omega_c z)\phi_{1x} - \omega_c x\phi_{1z}-\phi_{1t} = 0 \quad \text{for } y = 0 ,$$

and it is clear that conditions on the higher order terms could also be obtained if desired. In a similar fashion the kinematic free surface condition can be derived from (2.12); the lowest order term in $\beta$ yields this condition in the form:

$$(3.11)\qquad \phi_{1y} - (s_c+\omega_c z)\eta_{1x} +\omega_c x\eta_{1z}+\eta_{1t} = 0 \quad \text{for } y = 0 .$$

We turn next to the derivation of the linearized boundary conditions on the ship's hull. In view of (3.6) and (3.7), the transformation formulas (2.14) can be put in the form

$$(3.12)\qquad \begin{cases} x' = x+\beta\theta_{31}(y-y_c') -\beta\theta_{21}z \\ y' = y-\beta y_1+\beta\theta_{11}z-\beta\theta_{31}x \\ z' = z+\beta\theta_{21}x-\beta\theta_{11}(y-y_c') \end{cases}$$

when terms involving second and higher powers of $\beta$ are rejected. Consequently, the equation (2.16) of the ship's hull, up to terms in $\beta^2$, can be written as follows

$$z+\beta\theta_{21}x-\beta\theta_{11}(y-y_c')-\beta h[x+\beta\theta_{31}(y-y_c')-\beta\theta_{21}z, y-\beta y_1+\beta\theta_{11}z-\beta\theta_{31}x] = 0,$$

and, upon expanding the function $h$, the equation becomes

$$(3.13) \qquad \bar{z}+\beta\theta_{21}x-\beta\theta_{11}(y-y_c')-\beta h(x,y)+\ldots = 0 ,$$

the dots representing higher order terms in $\beta$. We can now obtain the kinematic boundary condition for the hull by inserting the left hand side of (3.13) for the function $f$ in (2.11); the result is

$$(3.14) \qquad \begin{cases} \omega_0 = 0 \\ \phi_{1z} = -s_c(\theta_{21}-h_x)+x_1+\theta_{21}x-\theta_{11}(y-y_c') \end{cases}$$

when the terms of zero and first order only are taken into account. It is clear that these conditions are to be satisfied over the domain A of the x,y-plane that is covered by the projection of the hull on the plane when the ship is in the rest position of equilibrium. As was mentioned earlier, it turns out that $\omega_0 = 0$, i.e. that the angular velocity about the z-axis of the c.g. of the ship in its course must be small of first order, or, as it could also be put, the curvature of the ship's course must be small since the speed in the course is finite. The quantity $s_1(t)$ in (3.4) thus yields the oscillation of the ship relative to the x-axis.

It should also be noted that if we use $z = -\beta h(x,y)$ we find, corresponding to (3.14), that

$$\phi_{1z} = -s_o(\theta_{21}+h_x) + (\omega_1+\theta_{21})x - \theta_{11}(y-y_c') \ .$$

This means that A must be regarded as two sided, and that the last equation is to be satisfied on the side of A which faces the negative z-axis. The last equation and (3.14) implies that $\phi$ may be discontinuous at the disk A.

The next step in the procedure is to substitute the developments with respect to $\beta$, (3.2) - (3.7), in the conditions for the ship's hull given by (2.18) and (2.22). Let us begin with the integral $\int_S p\vec{n}\, ds$ which appears in (2.18). In this integral S is the immersed surface of the hull, $\vec{n}$ is the inward unit normal to this surface and p is the pressure on it which is to be calculated from (2.[illegible]). With respect to the o-x,y,z coordinate system the equations of the symmetrical halves of the hull are

$$(3.15)\qquad \begin{aligned} S_1:- &\qquad z = H_1(x,y,t;\beta) = f_1 + f_2 \\ S_2:- &\qquad z = H_2(x,y,t;\beta) = -f_1 + f_2 \end{aligned}$$

where

$$f_1 = \beta h + \beta^2[\theta_{31}(y-y_c')h_x-(\theta_{31}x+y_1)h_y] + O(\beta^3) \tag{3.16}$$

$$f_2 = -\beta\theta_{21}x + \beta\theta_{11}(y-y_c') + O(\beta^2) .$$

We can now write

$$\int_S p \vec{n} \, ds = \int_{S_1} p \vec{n}_1 ds_1 + \int_{S_2} p \vec{n}_2 ds_2$$

in which $n_1$ and $n_2$ are given by

$$n_1 = \frac{H_{1x}\vec{i}+H_{1y}\vec{j}-\vec{k}}{\sqrt{1+H_{1x}^2+H_{1y}^2}} \quad ; \qquad \vec{n}_2 = \frac{-H_{2x}\vec{i}-H_{2y}\vec{j}+\vec{k}}{\sqrt{1+H_{2x}^2+H_{2y}^2}} \quad .$$

We can also write

$$\int_S p \vec{n} \, ds = -\rho g \int_S y \vec{n} \, ds + \int_S p_1 \vec{n} \, ds =$$

$$= -\rho g \int_S y \vec{n} \, ds + \int_{S_1} p_1 \vec{n}_1 ds_1 + \int_{S_2} p_1 \vec{n} \, ds_2$$

where $p_1$, from (2.9), is

$$p_1 = -\rho[\tfrac{1}{2}(\text{grad } \phi)^2 + (s+\omega z)\phi_x - x\omega\phi_z - \phi_t] . \tag{3.17}$$

If $S_o$ is the hull surface below the xz-plane, the surface area $S_o-S$ is of order $\beta$ and in this area each of the quantities $y$, $H_1$, $H_2$ is of order $\beta$. Hence

$$-\int_S y\vec{n}\,ds = -\int_{S_o} y\vec{n}\,ds + (\vec{i}+\vec{j})\,O(\beta^3) + \vec{k}\,O(\beta^2)$$

From the divergence theorem

$$-\int_{S_o} y\vec{n}\,ds = V\vec{j}$$

where $V$ is the volume bounded by $S_o$ and the xz-plane. With an accuracy of order $\beta^3$, $V$ is given by

$$V = 2\beta\int_A h\,dA - \int_B \beta(y_1+\theta_{31}x)\,dB = 2\beta\int_A h\,dA - 2\beta^2\int_L (y_1+x\theta_{31})h\,dx.$$

Here $A$ is the projection of the hull on the vertical plane when the hull is in the equilibrium position, $B$ is the equilibrium water line area, and $L$ is the projection of the equilibrium water line on the x-axis.

If $W_1, W_2$ are the respective projections of the immersed surfaces $S_1$, $S_2$ on the xy-plane we have

$$\int_S p_1 \vec{n}\, d = \vec{i}\left\{\int_{W_1} p_1(x,y,H_1;t)H_{1x}\,dW_1 - \int_{W_2} p_1(x,y,H_2,t)H_{2x}\,dW_2\right\}$$
$$+ \vec{j}\left\{\int_{W_1} p_1(x,y,H_1;t)H_{1y}\,dW_1 - \int_{W_2} p_1(x,y,H_2;t)H_{2y}\,dW_2\right\}$$
$$- \vec{k}\left\{\int_{W_1} p_1(x,y,H_1;t)\,dW_1 - \int_{W_2} p_1(x,y,H_2;t)\,dW_2\right\} .$$

Neither $W_1$ nor $W_2$ is equal to A. Each of the differences $W_1$-A, $W_2$-A is, however, an area of order $\beta$. From this and the fact that each of p, $H_{1x}$, $H_{1y}$, $H_{2x}$, $H_{2y}$ is of order $\beta$, it follows that

$$\int_S p_1 \vec{n}\, ds = \vec{i}\left\{\iint_A [p_1(x,y,H_1;t)H_{1x} - p_1(x,y,H_2;t)H_{2x}]\,dA + O(\beta^3)\right\}$$
$$+ \vec{j}\left\{\iint_A [p_1(x,y,H_1;t)H_{1y} - p_1(x,y,H_2;t)H_{2y}]\,dA + O(\beta^3)\right\}$$
$$- \vec{k}\left\{\iint_A [p_1(x,y,H_1;t) - p_1(x,y,H_2;t)]\,dA + O(\beta^2)\right\} .$$

It was pointed out above that $\phi$ may be discontinuous on A. Hence from (3.17), (3.2), (3.4)

$$(3.19)\qquad \begin{cases} p_1(x,y,H_1;t) = \rho\beta(\phi_{1t} - s_o\phi_{1x})^+ + O(\beta^2) \\ \\ p_1(x,y,H_2;t) = \rho\beta(\phi_{1t} - s_o\phi_{1x})^- + O(\beta^2) \end{cases} .$$

Here the + and - superscripts denote values at the positive and negative sides of the disk A whose positive side is regarded as the side which faces the positive z-axis. If we substitute the developments of $H_{1x}$, $H_{1y}$, $H_{2x}$, $H_{2y}$, and (3.19) in (3.18), then collect the previous results, we find

$$
\begin{aligned}
\int_S p\,\vec{n}\,ds = {} & \vec{i}\left\{\rho\beta^2\int_A [(h_x-\theta_{21})(\phi_{1t}-s_o\phi_{1x})^+ + (h_x+\theta_{21})(\phi_{1t}-s_o\phi_{1x})^-]\,dA + O(\beta^3)\right\} \\
& + \vec{j}\left\{\begin{aligned} & 2\rho g\beta\int_A h\,dA - 2\rho g\beta^2\int_L (y_1+x\theta_{31})h\,dx \\ & +\rho\beta^2\int_A [(h_y+\theta_{11})(\phi_{1t}-s_o\phi_{1x})^+ + (h_y-\theta_{11})(\phi_{1t}-s_o\phi_{1x})^-]\,dA + O(\beta^3)\end{aligned}\right\} \\
& - \vec{k}\left\{\rho\beta\int_A [(\phi_{1t}-s_o\phi_{1x})^+ - (\phi_{1t}-s_o\phi_{1x})^-]\,dA + O(\beta^2)\right\} .
\end{aligned} \tag{3.20}
$$

The integral $\int_S p(\vec{r}-y_c\vec{j})\times\vec{n}\,ds$ which appears in (2.22) can be written

$$
\begin{aligned}
\int_S p(\vec{r}-y_c\vec{j})\times\vec{n}\,ds = {} & -\rho g\int_S y(\vec{r}-y_c\vec{j})\times\vec{n}\,ds \\
& + \int_{S_1} p_1(\vec{r}-y_c\vec{j})\times\vec{n}_1\,dS_1 \\
& + \int_{S_2} p_1(\vec{r}-y_c\vec{j})\times\vec{n}_2\,dS_2 \qquad .
\end{aligned}
$$

If we use the same procedure as was used above for the expansion of $\int_S p\,\vec{n}\,ds$ we find

$$\int_S p(\vec{r}-y_c\vec{j})\times\vec{n}\,ds$$

$$= -\vec{i}\left\{\rho\beta\int_A[(y-y_c)(\phi_{1t}-s_o\phi_{1x})^{+}-(y-y_c)(\phi_{1t}-s_o\phi_{1x})^{-}]dA+O(\beta^2)\right\}$$

$$+\vec{j}\left\{\rho\beta\int_A[x(\phi_{1t}-s_o\phi_{1x})^{+}-x(\phi_{1t}-s_o\phi_{1x})^{-}]dA+O(\beta^2)\right\}$$

$$+\vec{k}\left\{\begin{array}{l} 2\rho g\beta\int_A xh\,dA-2\rho g\beta^2\theta_{31}\int_A(y-y_c')h\,dA \\ \quad -2\rho g\beta^2 y_1\int_L xh\,dx-2\rho g\beta^2\theta_{31}\int_L x^2h\,dx \\ +\rho\beta^2\int_A[x(h_y+\theta_{11})(\phi_{1t}-s_o\phi_{1x})^{+}+x(h_y-\theta_{11})(\phi_{1t}-s_o\phi_{1x})^{-}]dA \\ -\rho\beta^2\int_A[(y-y_c)(h_x-\theta_{21})(\phi_{1t}-s_o\phi_{1x})^{+}+(y-y_c)(h_x+\theta_{21})(\phi_{1t}-s_o\phi_{1x})^{-}]dA \\ \qquad\qquad + O(\beta^3) \end{array}\right.$$

We now assume that the propeller thrust $\vec{T}$ is of order $\beta^2$ and is directed parallel to the x' axis: that is

$$\vec{T} = \beta^2\, r\, \vec{i}\,'$$

where $\vec{i}'$ is the unit vector along the x'-axis. We also assume that $\vec{T}$ is applied at a point in the longitudinal plane of symmetry of the ship $\ell$ units below the center of mass. It then follows that

$$\vec{T} = \beta^2 T\vec{i} + O(\beta^3) \tag{3.22}$$

and

$$\begin{aligned}(\vec{R}_T - \vec{R}_c)\times\vec{T} &= -\ell\,\vec{j}\times\vec{T} \\ &= \ell\,\beta^2 T\vec{k} + O(\beta^3)\ .\end{aligned} \tag{3.23}$$

The mass of the ship is of order $\beta$. If we write $m = m_1\beta$ and expand the left hand side of (2.16) in powers of $\beta$ it becomes

$$\begin{aligned}&\vec{i}[m_1\beta\dot{s} + m_1\beta^2\dot{s}_1 + O(\beta^3)] + \vec{j}[m_1\beta^2\ddot{y}_1 + O(\beta^3)] - \vec{k}[O(\beta^3)] \\ &= \int_S p\,\vec{n}\,ds + \vec{T} - m_1\beta g\vec{j}\ .\end{aligned} \tag{3.24}$$

The expansion of the left hand side of (2.22) gives

$$\begin{aligned}&\vec{i}[O(\beta^2)] + \vec{j}[O(\beta^2)] + \vec{k}[I_{31}\beta^2\ddot{\theta}_{31} + O(\beta^3)] \\ &= \int_S p(\vec{r} - y_c\vec{j})\times\vec{n}\,ds + (\vec{R}_T - \vec{R}_c)\times\vec{T}\end{aligned} \tag{3.25}$$

where $\beta I_{31}$ is the moment of inertia of the ship about the axis which is perpendicular to the longitudinal plane of symmetry of the ship and which passes through the center of mass.

If we replace the pressure integrals and thrust terms in the last two equations by (3.20), (3.21), (3.22), (3.23), and then equate the coefficients of like powers of $\beta$ in (3.24) and (3.25) we obtain the following linearized equations of motion of the ship. From the first order terms we find

(3.26) $$\dot{s}_0 = 0$$

(3.27) $$2\rho g \int_A \beta h\, dA = M_1 \beta g$$

(3.28) $$\int_A x\beta h\, dA = 0$$

(3.29) $$\int_A [(\varphi_{1t} - s_0\varphi_{1x})^+ - (\varphi_{1t} - s_0\varphi_{1x})^-]\, dA = 0$$

(3.30) $$\int_A [x(\varphi_{1t} - s_0\varphi_{1x})^+ - x(\varphi_{1t} - s_0\varphi_{1x})^-]\, dA = 0$$

(3.31) $$\int_A [(y - y_c')(\varphi_{1t} - s_0\varphi_{1x})^+ - (y - y_c')(\varphi_{1t} - s_0\varphi_{1x})^-]\, dA = 0$$

or by (3.29)

(3.32) $$\int_A [y(\varphi_{1t} - s_0\varphi_{1x})^+ - y(\varphi_{1t} - s_0\varphi_{1x})^-]\, dA = 0 .$$

From the second order terms we find

$$M_1\dot{s}_1 = \rho\int_A[(h_x-\theta_{21})(\phi_{1t}-s_0\phi_{1x})^+ + (h_x+\theta_{21})(\phi_{1t}-s_0\phi_{1x})^-]dA+T$$

(3.33)

$$= \rho\int_A[h_x(\phi_{1t}-s_0\phi_{1x})^+ + h_x(\phi_{1t}-s_0\phi_{1x})^-]dA+T$$

$$M_1\ddot{y}_1 = -2\rho g\int_L(y_1+x\theta_{31})hdx$$

$$+\rho\int_A[(h_y+\theta_{11})(\phi_{1t}-s_0\phi_{1x})^+ + (h_y-\theta_{11})(\phi_{1t}-s_0\phi_{1x})^-]dA$$

(3.34) $$= -2\rho g\int_L(y_1+x\theta_{31})hdx$$

$$+\rho\int_A[h_y(\phi_{1t}-s_0\phi_{1x})^+ + h_y(\phi_{1t}-s_0\phi_{1x})^-]dA$$

$$I_{31}\ddot{\theta}_{31} = -2\rho g\theta_{31}\int_A(y-y_c')hdA-2\rho gy_1\int_L xhdx$$

$$-2\rho g\theta_{31}\int_L x^2hdx+\nu T$$

$$+\rho\int_A[x(h_y+\theta_{11})(\phi_{1t}-s_0\phi_{1x})^+ + x(h_y-\theta_{11})(\phi_{1t}-s_0\phi_{1x})^-]dA$$

$$-\rho\int_A[(y-y_c')(h_x-\theta_{21})(\phi_{1t}-s_0\phi_{1x})^+ + (y-y_c')(h_x+\theta_{21})(\phi_{1t}-s_0\phi_{1x})^-]dA$$

or by (3.30), (3.31)

$$I_{31}\ddot{\theta}_{31} = -2\rho g\theta_{31}\int_A (y-y_c')h\,dA - 2\rho g y_1\int_L xh\,dx$$

(3.35)

$$-2\rho g\theta_{31}\int_L x^2 h\,dx + \ell T$$

$$+\rho\int_A [xh_y-(y-y_c')h_x][(\varphi_{1t}-\dot{s}_c\varphi_{1x})^+ + (\varphi_{1t}-\dot{s}_c\varphi_{1x})^-]\,dA \;.$$

Equation (3.26) states that the motion in the x-direction is a small oscillation relative to a motion with uniform speed $\dot{s}_c$ = cons't. Equation (3.27) is an expression of Archimedes' law: the rest position of equilibrium must be such that the weight of the water displaced by the ship just equals the weight of the ship. The center of buoyancy of our ship is in the plane of symmetry, and equation (3.28) is an expression of the second law of equilibrium of a floating body; namely that the center of buoyancy for the equilibrium position is in the same vertical line, the y'-axis, as the center of gravity of the ship.

The function $\varphi_1$ must satisfy

$$\varphi_{1xx} + \varphi_{1yy} + \varphi_{1zz} = 0$$

in the domain D - A where D is the half space $y < 0$, and A is the plane disk defined by the projection of the submerged hull on the xy-plane when the ship is in the equilibrium position. We assume that A intersects the xz-plane. The boundary conditions at each side of A are

$$
(3.36)\begin{cases} \phi_{1z}^{+} = s_o(h_x-\theta_{21})+(\omega_1+\dot{\theta}_{21})x - \dot{\theta}_{11}(y-y'_c) \\ \\ \phi_{1z}^{-} = -s_o(h_x+\theta_{21}) + (\omega_1+\dot{\theta}_{21})x - \dot{\theta}_{11}(y-y'_c) \end{cases} .
$$

The boundary condition at $y = 0$ is found by eliminating $\eta_1$ from (3.10) and (3.11). Since $\omega_o = 0$ these equations are

$$
g\eta_1 + s_o\phi_{1x} - \phi_{1t} = 0
$$

$$
\phi_{1y} - s_o\eta_{1x} + \eta_{1t} = 0
$$

and they yield

$$
(3.37) \qquad s_o^2\phi_{1xx} - 2s_o\phi_{1xt} + g\phi_{1y} + \phi_{1tt} = 0
$$

for $y = 0$. The boundary conditions (3.36) and (3.37) show that $\phi_1$ depends on $\omega_1(t)$, $\theta_{11}(t)$ and $\theta_{21}(t)$. The potential problem can theoretically be solved for $\phi_1$ in the form

$$
\phi_1 = \phi_1[x,y,z,t;\ \omega_1(t),\theta_{11}(t),\theta_{21}(t)]
$$

without using (3.29), (3.30), (3.31). The significance of this has already been discussed in Sec. 1 in relation to equation (1.14). The general procedure to be followed in solving all problems was all discussed there.

The general potential problem defined above will be the subject of a separate study. The remainder of this paper is concerned with the special case of a ship which moves along a straight course into waves whose crests are at right angles to the course. For this case there are surging, heaving and pitching motions, but $\theta_1 = 0$, $\theta_2 = 0$, $\omega = 0$ and the potential function $\varphi$ is an even function of z. Under these conditions the equations of motion are more simple. They are

$$(3.38) \qquad M_1\ddot{s}_1 = 2\rho\int_A h_x(\varphi_{1t} - s\,\varphi_{1x})\,dA + T$$

$$(3.39) \qquad M_1\ddot{y}_1 = -2\rho g y_1\int_L h\,dx - 2\rho g\theta_{31}\int_L xh\,dx + 2\rho\int_A h_y(\varphi_{1t} - s\,\varphi_{1x})\,dA$$

$$(3.40) \qquad I_{31}\ddot{\theta}_{31} = -2\rho g\theta_{31}\int_A (y - y_c')h\,dA - 2\rho g y_1\int_L xh\,dx$$

$$-2\rho g\theta_{31}\int_L x^2 h\,dx + \ell T$$

$$+2\rho\int_A [xh_y - (y - y_c')h_x](\varphi_{1t} - s\,\varphi_{1x})\,dA.$$

It will be shown in the next section that an explicit integral representation can be found for the corresponding potential function and that this leads to integral representations for $s_1$, $y_1$ and $\theta_{31}$.

4. Method of solution of the problem of pitching and heaving of a ship in a sea-way having normal incidence.

In this section we derive a method of solution of the problem of calculating the pitching, surging, and heaving motions in a sea-way consisting of a train of waves moving at right angles to the course of the ship, which is assumed to be a straight line (i.e. $\omega \equiv 0$). The propeller thrust is assumed to be a constant vector.

The harmonic function $\phi_1$ and the surface elevation $\eta_1$ therefore satisfy the following free surface conditions (cf. (3.10) and (3.11), with $\omega_o = 0$):

$$(4.1) \qquad \left.\begin{aligned} g\eta_1 + s_o\phi_{1x} - \phi_{1t} &= 0 \\ \phi_{1y} - s_o\eta_{1x} + \eta_{1t} &= 0 \end{aligned}\right\} \quad \text{at } y = 0 .$$

The kinematic condition arising from the hull of the ship is (cf. (3.14) with $\theta_{21} = \theta_{11} = \omega_1 = 0$):

$$(4.2) \qquad \phi_{1z} = s_o h_x .$$

Before writing down other conditions, including conditions at $\infty$, we express $\phi_1$ as a sum of two harmonic functions, as follows:

$$(4.3) \qquad \phi_1(x,y,z;t) = \chi_o(x,y,z) + \chi_1(x,y,z;t).$$

Here $\chi_o$ is a harmonic function independent of t which is also an even function of z. We now suppose that the motion of the ship is a steady simple harmonic motion in the time when observed from the moving coordinate system o-x,y,z. (Presumably such a state would result after a long time upon starting from rest under a constant propeller thrust.) Consequently we interpret $\chi_1(x,y,z)$ as the disturbance caused by the ship, which therefore dies out at $\infty$; while $\chi_1(x,y,z;t)$ represents a train of simple harmonic plane waves covering the whole surface of the water. Thus $\chi_1$ is given, with respect to the fixed coordinate system O-X,Y,Z, by the familiar formula

$$\chi_1 = C e^{\frac{\sigma^2}{g}Y} \sin\left(\sigma t + \frac{\sigma^2}{g} X + p\right),$$

with $\sigma$ the frequency of the waves. In the o-x,y,z system we have, therefore:

$$(4.4) \qquad \chi_1(x,y,z;t) = C e^{\frac{\sigma^2}{g} y} \sin\left[\frac{\sigma^2}{g} x + \left(\sigma + \frac{s_o \sigma^2}{g}\right)t + p\right].$$

We observe that the frequency, relative to the ship, is increased above the value $\sigma$ if $s_o$ is positive - i.e. if the ship is heading into the waves - and this is, of course, to be expected. With this choice of $\chi_1$, it is easy to verify that $\chi_o$ satisfies the following conditions:

$$(4.5) \qquad s_o^2\chi_{oxx} + g\chi_{oy} = 0 \quad \text{at } y = 0 ,$$

obtained after eliminating $\eta_1$ from (4.1), and

$$(4.6) \qquad \chi_{oz} = s_o h_x \quad \text{on } A,$$

with A, as above, the projection of the ship's hull (for $z > 0$) on its vertical mid-section. In addition, we require that $\chi_o \rightarrow 0$ at $\infty$.

It should be remarked at this point that the classical problem concerning the waves created by the hull of a ship, first treated by Michell [8], Havelock [2], and many others, is exactly the problem of determining $\chi_o$ from the conditions (4.5) and (4.6). Afterwards, the insertion of $\phi_1 \equiv \chi_o$ in (3.35), with $\dot{s}_1 = 0$, $\phi_{1t} \equiv 0$, leads to the formula for the wave resistance of the ship — i.e. the propeller thrust T is determined. Since $y_1$ and $\theta_3$ are independent of the time in this case, one sees that the other dynamical equations, (3.39) and (3.40), yield the displacement of the c.g. relative to the rest position of equilibrium (the so-called heave), and the longitudinal tilt angle (called the pitching angle). However, in the literature cited, the latter two quantities seem to be taken as zero, which implies that appropriate constraining forces would be needed to hold the ship in such a position relative to the water. However, the main quantity of interest is the wave resistance, and it is not affected (in the first order theory, at least) by the heave and pitch.

We proceed to the determination of $\chi_0$, using a method different from the classical method and following, rather, a course which it is hoped can be generalized in such a way as to yield solutions in more difficult cases.

Suppose that we know the Green's function $G^*(\xi,\eta,\zeta;x,y,z)$ such that $G^*$ is a harmonic function for $\eta < 0$, $\zeta > 0$ except at $(x,y,z)$ where it has the singularity $1/r$; and $G^*$ satisfies the boundary conditions

$$(4.7) \qquad G^*_{\xi\xi} + kG^*_{\eta} = 0 \qquad \text{on} \quad \eta = 0$$

$$G^*_{\zeta} = 0 \qquad \text{on} \quad \zeta = 0$$

where $k = g/s_0^2$. Let $\Sigma$: denote the half-plane $\eta = 0$, $\zeta > 0$; and let $\Omega$ denote the half-plane $\zeta = 0$, $\eta < 0$. Green's formula shows that

$$4\pi\chi_0 = -\iint_{\Sigma:} \chi_1 G^*_{\eta} d\xi d\zeta + \iint_{\Sigma:} \chi_{0\eta} G^* d\xi d\zeta - \iint_{\Omega} \chi_{0\zeta} G^* d\xi d\eta \,.$$

Then, since

$$-\iint_{\Sigma:} \chi_0 G^*_{\eta} d\xi d\zeta + \iint_{\Sigma:} \chi_{0\eta} G^* d\xi d\zeta = \frac{1}{k}\iint_{\Sigma.} (\chi_0 G^*_{\xi\xi} - \chi_{0\xi\xi} G^*) d\xi d\zeta$$

$$= \frac{1}{k}\iint_{\Sigma:} \frac{\partial}{\partial\xi} (\chi_0 G^*_{\xi} - \chi_{0\xi} G^*) d\xi d\zeta$$

$$= 0 \,,$$

we have an explicit representation of the solution in the form

$$\chi_0(x,y,z) = -\frac{1}{4\pi}\iint_{\Omega} \chi_{0\zeta} G^* \, d\xi \, d\eta \quad , \text{ or}$$

$$(4.8) \qquad \chi_0(x,y,z) = -\frac{s_0}{4\pi}\iint_{A} h_\xi(\xi,\eta)\, G^*(\xi,\eta,0\,;x,y,z)\, d\xi \, d\eta,$$

upon using (4.6).

In order to determine $G^*$ consider the Green's function $G(\xi,\eta,\zeta;x,y,z)$ for the half space $\eta < 0$ which satisfies

$$G_{\xi\xi} + kG_\eta = 0$$

on $\eta = 0$. This function can be written as

$$G = \frac{1}{r_1} - \frac{1}{r_2} + g$$

where

$$\frac{1}{r_1} = \frac{1}{\sqrt{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2}}$$

$$\frac{1}{r_2} = \frac{1}{\sqrt{(\xi-x)^2+(\eta+y)^2+(\zeta-z)^2}}$$

and $g$ is a potential function in $\eta < 0$ which satisfies

$$g_{\xi\xi} + kg_{\eta} = 2k \frac{\partial}{\partial y} \frac{1}{\sqrt{(\xi-x)^2+y^2+(\zeta-z)^2}}$$

on $\eta = 0$. The well-known formula

$$2k \frac{\partial}{\partial y} \frac{1}{\sqrt{(\xi-x)^2+y^2+(\zeta-z)^2}} = 2k \int_0^\infty pe^{py} J_0 [p\sqrt{(\xi-x)^2+(\zeta-z)^2}]dp,$$

in which the Bessel function $J_0$ can be expressed as

$$J_0 [p\sqrt{(\xi-x)^2+(\zeta-z)^2}] = \frac{2}{\pi} \int_0^{\pi/2} \cos[p(\xi-x)\cos\theta]\cos[p(\zeta-z)\sin\theta]d\theta,$$

allows us to write

$$g_{\xi\xi} + kg_{\eta} = \frac{4k}{\pi} \int_0^\infty \int_0^{\pi/2} pe^{py} \cos[p(\xi-x)\cos\theta]\cos[p(\zeta-z)\sin\theta]d\theta dp$$

for $\eta = 0$ and $y < 0$. It is now easy to see that

$$g_{\xi\xi} + kg_{\eta} = \frac{4k}{\pi} \int_0^\infty \int_0^{\pi/2} pe^{p(y+\eta)} \cos[p(\xi-x)\cos\theta][p(\zeta-z)\sin\theta]d\theta dp$$

is a potential function in $\eta < 0$ which satisfies the boundary condition. An interchange of the order of integration gives

$$g_{\xi\xi} + k g_{\eta} = \frac{4k}{\pi} \int_0^{\pi/2} d\theta\, \mathcal{R}_e \int_0^{\infty} p \cos[p(\zeta - z)\sin\theta]\, e^{p[(y+\eta)+i(\xi-x)\cos\theta]}\, dp$$

where $\mathcal{R}_e$ denotes the real part. If we think of p as a complex variable, the path from 0 to $\infty$ in the last result can be replaced by an equivalent path L:

$$g_{\xi\xi} + k g_{\eta} = \frac{4k}{\pi} \int_0^{\pi/2} d\theta\, \mathcal{R}_e \int_L p \cos[p(\zeta - z)\sin\theta]\, e^{p[(y+\eta)+i(\xi-x)\cos\theta]}\, dp .$$

Since the right hand side of this differential equation for g is expressed as a superposition of exponentials in $\xi$ and $\eta$, and since some freedom is allowed in the choice of L, it is evident that

$$g = \frac{4k}{\pi} \int^{\pi/2} d\theta\, \mathcal{R}_e \int_L \frac{p \cos[p(\zeta - z)\sin\theta]\, e^{p[(y+\eta)+i(\xi-x)\cos\theta]}}{kp - p^2\cos^2\theta}\, dp$$

provided L can be properly chosen. The path L, which will be fixed by a condition given below, must, of course, avoid the pole at $p = k/\cos^2\theta$.

It can now be seen that the function $G^*(\xi,\eta,\zeta;x,y,z) = G(\xi,\eta,\zeta;x,y,z) + G(\xi,\eta,\zeta;x,y,-z)$ satisfies all of the conditions imposed on the Green's function employed in (4.8): the sum on the right has the proper singularity in $\eta<0$, $\zeta>0$, it satisfies the boundary condition (4.7) and

$$G_\zeta(\xi,\eta,\zeta;x,y,z) + G_\zeta(\xi,\eta,\zeta;x,y,-z)$$

is zero at $\zeta = 0$. Therefore

$$G^*]_{\zeta=0} = 2\left[\frac{1}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}} - \frac{1}{\sqrt{(\xi-x)^2+(\eta+y)^2+z^2}}\right.$$

$$\left. + \frac{8k}{\pi}\int_0^{\pi/2} d\theta \, \mathcal{R}e \int_L \frac{\cos(pz\sin\theta)e^{p[(y+\eta)+i(\xi-x)\cos\theta]}\,dp}{k-p\cos^2\theta}\right] .$$

The substitution of this in (4.8) gives finally

$$\chi_c(x,y,z) = -\frac{s_c}{2\pi}\iint_A h_\xi(\xi,\eta)\left\{\frac{1}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}} - \frac{1}{\sqrt{(\xi-x)^2+(\eta+y)^2+z^2}}\right\} d\xi\, d\eta$$

$$-\frac{2ks_c}{\pi^2}\iint_A h_\xi(\xi,\eta)\left\{\int_0^{\pi/2} d\theta\, \mathcal{R}e\int_L \frac{\cos(pz\sin\theta)e^{p[(y+\eta)+i(\xi-x)\cos\theta]}\,dp}{k-p\cos^2\theta}\right\} d\xi\, d\eta.$$

A condition imposed on $\chi_c(x,y,z)$ is that $\chi_c(x,y,z) \to 0$ as $x \to +\infty$. This condition is satisfied if we take L to be the path shown in Fig. 4.1.

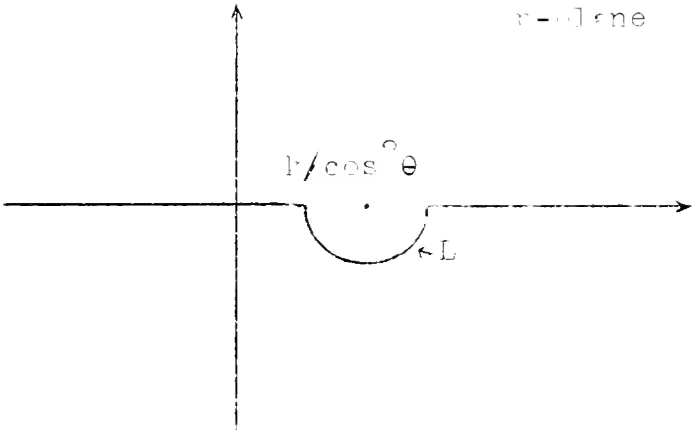



Fig. 4.1.

The Path L in the p-plane

The function $\phi_1$ is given by

$$\phi_1 = \chi_1 + \chi_0 = C\, e^{\frac{\sigma^2 y}{g}} \sin\left[\frac{\sigma^2 x}{g} + \left(\sigma + \frac{s_1 \sigma^2}{g}\right)t + p\right] + \chi_0$$

and therefore

$$(4.9) \quad \phi_{1t} - s\,\phi_{1x} = C\sigma e^{\frac{\sigma^2 y}{g}} \cos\left[\frac{\sigma^2 x}{g} + \left(\sigma + \frac{s_1\sigma^2}{g}\right)t + p\right] - s\,\chi_{0x}$$

If this is substituted in the equation for the surge we have

$$M_1 \dot{s}_1 = 2\rho C\sigma \iint_A h_x e^{\frac{\sigma^2 y}{g}} \cos\left[\frac{\sigma^2 x}{g} + \left(\sigma + \frac{s_1\sigma^2}{g}\right) t + p\right] dx\,dy$$

$$- 2\rho s_1 \iint_A h_x \chi_x \, dx\, dy + f \, .$$

The last equation shows that in order to keep $s_1$ bounded for all t we must take

$$(4.10) \qquad T = 2\rho s_o \iint_A h_x \chi_{ox} \, dx\,dy$$

where

$$\chi_{ox}(x,y,0) = -\frac{s_o}{2\pi} \iint_A h_\xi(\xi,\eta) \left\{ \frac{(\xi-x)}{[(\xi-x)^2+(\eta-y)^2]^{3/2}} - \frac{(\xi-x)}{[(\xi-x)^2+(\eta+y)^2]^{3/2}} \right\} d\xi\, d\eta$$

$$+ \frac{2s_o}{\pi^2} \iint_A h_\xi(\xi,\eta) \left\{ \int_0^{\pi/2} d\theta \, \mathcal{R}_e \int_L \frac{igp \cos\theta \; e^{p[(y+\eta)+i(\xi-x)\cos\theta]} \, dp}{g-s_o^2 \, p \cos^2\theta} \right\} d\xi\, d\eta.$$

Equation (4.10) gives the thrust necessary to maintain the speed $s_o$, or inversely it gives the speed $s_o$ which corresponds to a given thrust. The integral in (4.10) is called the wave resistance integral. As one sees, it does not depend on the seaway. The integral can be expressed in a more simple form as follows.

The function $\chi_{ox}(x,y,o)$ is a sum of integrals of the type

$$\iint_A h_\xi(\xi,\eta) \; f(\xi,\eta;x,y) \; d\xi\, d\eta.$$

If an integral of this type is substituted in the wave resistance integral we have

$$\iint_A \iint_A h_x(x,y)h_\xi(\xi,\eta)f(\xi,\eta;x,y)\, d\xi d\eta dx dy = I$$

say. This is the same as

$$\iint_A \iint_A h_\xi(\xi,\eta)h_x(x,y)f(x,y;\xi,\eta)\, dx dy d\xi d\eta = I$$

and we see that $I = 0$ if

$$f(\xi,\eta;x,y) = - f(x,y;\xi,\eta) .$$

Therefore

$$T = \frac{4\rho s^2}{\pi^2} \iint_A \iint_A h_x(x,y)h_\xi(\xi,\eta)\; f_1\; d\xi d\eta dx dy$$

where

$$f_1 = \int_0^{\pi/2} d\theta \,\mathcal{R}_e \int_L \frac{igp \cos\theta\, e^{p(y+\eta)} \cos[p(\xi - x) \cos\theta]}{g - s_c^2\, p \cos^2\theta}\, dp .$$

Since $\mathcal{R}_e \int_L$ is zero except for the residue from the integration along the semi-circular path about

$$\frac{g}{s_c^2 \cos^2\theta} = \frac{k}{\cos^2\theta} ,$$

we find from the evaluation of this residue that

$$f_1 = \frac{\pi g^2}{s_o^4} \int_0^{\pi/2} \sec^3\theta \, e^{k(y+\eta)\sec^2\theta} \cos\left[k(\xi - x)\cos\theta\right] d\theta \; .$$

Now if we define

$$P(\theta) = \iint_A h_x(x,y)\, e^{ky\sec^2\theta} \cos(kx\sec\theta)\, dxdy$$

$$Q(\theta) = \iint_A h_x(x,y)\, e^{ky\sec^2\theta} \sin(kx\sec\theta)\, dxdy$$

we can write

$$T = \frac{4\rho g^2}{\pi s_o^2} \int_0^{\pi/2} (P^2 + Q^2) \sec^3\theta \, d\theta \; .$$

This is the familiar formula of Michell for the wave resistance. The surge is given by

$$s_1 = \frac{2\rho C\sigma g}{(g\sigma + s_o\sigma^2)M_1} \iint_A h_x \, e^{\frac{\sigma^2 y}{g}} \left[\frac{\sigma^2}{g} x + \left(\sigma + \frac{s_o\sigma^2}{g}\right) t + p\right] dxdy$$

Hereafter we will suppose for simplicity that there is no coupling between (3.39) and (3.40), so that $\int_L xhdx = 0$. The substitution of (4.9) in (3.39) there gives the following equation for the heave.

$$M_1\ddot{y}_1+[2\rho g\int_L h dx]y_1 = 2\rho C\sigma\iint_A h_y e^{\frac{\sigma^2 y}{g}}\cos\left|\frac{\sigma^2 x}{g}+(\sigma+\frac{s_o\sigma^2}{g})t+p\right|dxdy$$

$$-2\rho\, s_o\iint_A h_y\chi_{ox}\,dxdy$$

The time independent part of $y_1$, the heave component of the trim, we denote by $y_1^*$; it is given by

$$(4.11)\qquad (g\int_L h\,dx)y_1^* = -s_o\iint_A h_y\chi_{ox}\,dxdy.$$

$y_1^*$ is the vertical displacement of the center of gravity of a ship moving in calm water from its rest position. The integral on the right hand side of (4.11) is even more difficult to evaluate than the wave resistance integral. As far as the authors are aware, the integral has not appeared in the literature.

The response of $y_1$ to the sea is given by

$$y_1^{**} = \frac{2\rho C\sigma\iint_A h_y e^{\frac{\sigma^2 y}{g}}\cos\left[\frac{\sigma^2 x}{g}+(\sigma+s_o\frac{\sigma^2}{g})\,t+p\right]dxdy}{2\rho g\int_L h dx - M_1(\sigma+\frac{s_o\sigma^2}{g})^2}.$$

For the case under consideration, the theory predicts that resonance in the heave occurs when

$$\sigma + \frac{s_o \sigma^2}{g} = \left[ \frac{2\rho g}{M_1} \int_L h dx \right]^{1/2} .$$

The equation for the pitching angle is

$$I_{31}\ddot{\theta}_{31} + 2\rho g \left[ \int_A (y-y_c')hdA + \int_L x^2 hdx \right] \theta_{31}$$

$$= 2\rho C\sigma \iint_A [xh_y-(y-y_c')h_x]\cos\left\{\frac{\sigma^2 x}{g} + (\sigma + \frac{s_o \sigma^2}{g})t + p\right\} dxdy$$

$$+ \ell T - 2\rho s_o \int_A [xh_y-(y-y_c')h_x] \chi_{ox} \, dA .$$

The time independent part of $\theta_{31}$, $\theta_{31}^*$, is given by

$$2\rho g \left[ \int_A (y-y_c')hdA + \int_L x^2 h \, dx \right] \theta_{31}^*$$

$$= \ell T - 2\rho s_o \int_A [xh_y-(y-y_c')h_x] \chi_{ox} \, dA$$

$$= (\ell - y_c')T - 2\rho s_o \int_A [xh_y - yh_x]\chi_{ox} \, dA.$$

The angle $\theta_{31}^*$ is called the angle of trim; it is the angular displacement of a ship which moves with the speed $s_o$ in calm water.

The response of $\theta_{31}$ to the sea is

$$\theta_{31}^{**} = \frac{2\rho C\sigma \iint_A [xh_y-(y-y_c')h_x] \cos\left\{\frac{\sigma^2 x}{g} + (\sigma+\frac{s_o\sigma^2}{g})t+p\right\} dxdy}{2\rho g\left[\int_A (y-y_c')hdA+\int_L x^2 hdx\right] - I_{31}(\sigma+\frac{s_o\sigma^2}{g})^2} .$$

and we see that the theory predicts resonance when

$$\sigma+\frac{s_o\sigma^2}{g} = \left\{\frac{2\rho g}{I_{31}}\left[\int_A (y-y_c')hdA + \int_L x^2 hdx\right]\right\}^{1/2} .$$

## Bibliography


[1] Haskind, M. D., The Hydrodynamic Theory of the Oscillation of a Ship in Waves (in Russian), Prik. Mat. i. Mek., vol. 10, no. 1 (1946).
Oscillation of a Ship on a Calm Sea (in Russian), Izv. Ak. Nauk, Ot. Tex. Nauk, 1946.
Translated by V. Wehansen as "Two Papers on the Hydrodynamic Theory of Heaving and Pitching of a Ship", Tech. Res. Bull. 1-12, Soc. Nav. Arch. and Mar. Eng. (New York), 1953.

[2] Havelock, T. H., The Wave-making Resistance of Ships, Proc. Roy. Soc. London (A), 81 (1909).

[3] Havelock, T. H., Wave Resistance Theory and its Application to Ship Problems, Soc. Nav. Arch. and Mar. Eng. (New York), 1950.

[4] John, F., On the Motion of Floating Bodies I, II, Comm. Pure and App. Math., Vol. I (1948) and Vol. III (1950).

[5] Krylov, A. N., A General Theory of the Oscillations of a Ship on Waves, Trans. Inst. Nav. Arch. (40), 1896.

[6] Lunde, J. K., Wigley, W. C. S., Calculated and Observed Wave Resistances for a Series of Forms of Fuller Midsection. Quart. Trans. Inst. Naval Arch. (London) 90, April, 1948.

[7] Lunde, J. K., On the Linearized Theory of Wave Resistance for Displacement Ships in Steady and Accelerated Motion, Soc. Nav. Arch. and Mar. Eng. (1951).

[8] Michell, J. H., The Wave Resistance of a Ship, Phil. Mag. 45 (1898).

[9] St. Denis, M., Weinblum, G., On the Motions of Ships at Sea. Soc. Nav. Arch. and Mar. Eng. (1950).

[10] Weinblum, G. P., Analysis of Wave Resistance, Report 710 issued by the David W. Taylor Model Basin (Washington, D. C.), Sept. 1950.

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